

CLAW-FREE MAXIMAL PLANAR GRAPHS

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1. Introduction and Terminology

A graph G is claw-free if it contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. Such graphs have been widely studied with respect to such other graph properties as matching (cf. Sumner [13, 14] and Las Vergnas [5]), perfection (cf. Parthasarathy and Ravindra [9]), vertex-packing (cf. Minty [7] and Sbihi [12]), Hamiltonian cycles and related questions on traversability (cf. Oberly and Sumner [8], Clark [1] and Kanetkar and Rao [4]) and reconstruction (cf. Ellingham, Pyber and Yu [2]).

A planar graph is said to be maximal planar (or a triangulation) if, given any imbedding of G in the plane, every face boundary is a triangle. We shall use the abbreviations MAXP and CFMAXP for the properties *maximal planar* and *claw-free maximal planar* respectively. (Recall that every maximal planar graph with at least three points is either the complete graph K_3 or else is 3-connected and thus it follows that such a graph has a *unique* imbedding in the plane.)

In Section 2 of this paper, we present a constructive characterization of the family of CFMAXP graphs. In particular, the characterization proceeds as follows. First it is shown that if G is a 3-connected claw-free planar graph, then $\maxdeg G \leq 6$. We then show that there are precisely 8 such graphs with maximum degree no greater than 5. If G is CFMAXP and has $\maxdeg G = 6$, then G must have separating triangles and we fix our attention on these next. A special kind of separating triangle, called a *separating 345-triangle*, turns out to be the key to the characterization. If G is CFMAXP and has separating triangles, but no separating 345-triangles, then G is 1 of precisely 7 graphs. Finally, if G has a separating 345-triangle, we show that G must belong an infinite family of graphs which can easily be described recursively.

In Section 3, we present some results on traversability in CFMAXP graphs and in Section 4, some results on matching for this family of graphs.

Throughout this paper, we write $u \sim v$ when points u and v of a graph are joined by a line. Also if F is a face of a planar graph G , we shall write ∂F for the boundary of F .

2. The Characterization

First it will be shown that every *arbitrary* 3-connected claw-free planar graph G has $\maxdeg G \leq 6$. (That is, the graph need not be *maximal planar*.)

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Theorem 2.1. If G is 3-connected claw-free and planar, then

- (a) $\maxdeg G \leq 6$, and
- (b) if v has degree 6 in G , then v lies on at least two separating triangles.

Proof. Suppose that point $v \in V(G)$ has $\deg v \geq 7$ and suppose the neighbors of v (in clockwise order) are $u_1, \dots, u_7, \dots, u_r$, where $r \geq 7$.

First suppose all faces at v are triangles. We claim that there is an i such that $u_i \sim u_{i+2}$, where the subscripts are taken modulo r . If $u_1 \sim u_3$ or $u_3 \sim u_5$ we are done, so suppose neither adjacency holds. Then, since there is no claw at v , we must have $u_1 \sim u_5$. But then again, since there is no claw at v , either $u_3 \sim u_5$ or $u_5 \sim u_7$ and the Claim is proved.

So, renumbering if necessary, we may suppose that $u_1 \sim u_3$. Since $G[v, u_2, u_{r-2}, u_r]$ is not a claw, $u_{r-2} \sim u_r$. But then by planarity, $G[v, u_2, u_4, u_{r-1}]$ is a claw, contrary to hypothesis.

So we may suppose at least one of the faces F at v is not a triangle. Without loss of generality, suppose lines vu_1 and vu_2 lie in ∂F .

1. First suppose $u_1 \not\sim u_2$. Then by claw-freedom, either $u_1 \sim u_3$ or $u_2 \sim u_3$.

1.1. Suppose that $u_2 \not\sim u_3$, so that $u_1 \sim u_3$. Then by claw-freedom, subgraph $G[u_4, \dots, u_7, \dots, u_r]$ is a complete graph and since $r \geq 7$ it follows that $G[v, u_4, u_5, u_6, u_7]$ is isomorphic to K_5 , contradicting the planarity of G via Kuratowski's Theorem.

1.2. So suppose $u_2 \sim u_3$. Then by claw-freedom, either $u_4 \sim u_1$ or $u_4 \sim u_2$. First, suppose $u_4 \sim u_1$. Then by claw-freedom, $G[v, u_5, u_6, u_7, \dots, u_r]$ is complete and since G is planar, once again by Kuratowski's Theorem, we have $r = 7$ and $G[v, u_5, u_6, u_7]$ is complete. But then $G[v, u_1, u_2, u_6]$ is a claw, a contradiction.

So suppose that $u_4 \not\sim u_1$ and hence $u_4 \sim u_2$. Then $G[v, u_5, u_6, u_7, \dots, u_r]$ is complete by claw-freedom, and hence $G[v, u_1, u_2, u_6]$ is a claw, again a contradiction.

2. So suppose that $u_1 \sim u_2$. Thus since ∂F contains at least 4 points, we have that $\{u_1, u_2\}$ is a 2-cut in G , contradicting the assumption that G is 3-connected.

This completes the proof of (a).

In order to prove (b), let us suppose that $\deg v = 6$ and as above, let the neighbors of v in clockwise order be u_1, \dots, u_6 . By claw-freedom, we may assume that either $u_1 \sim u_2$ or $u_1 \sim u_3$.

1. Suppose $u_1 \sim u_3$. Then by claw-freedom, $u_4 \sim u_6$ and we have point v lying on 2 separating triangles (vu_1u_3v and vu_4u_6v) as claimed.

2. So suppose that $u_1 \not\sim u_3$ and hence $u_1 \sim u_2$. By symmetry, we may also suppose that $u_2 \not\sim u_4$, $u_3 \not\sim u_5$, $u_4 \not\sim u_6$, $u_5 \not\sim u_1$ and $u_6 \not\sim u_2$. But then $G[v, u_1, u_3, u_5]$ is a claw, contrary to hypothesis. ■

We then have the following immediate corollary.

Corollary 2.2. If G is a 3-connected claw-free planar graph with no separating triangle, then $\maxdeg G \leq 5$. ■

We are now prepared to find all claw-free maximal planar graphs containing no separating triangle.

Figure 1

Theorem 2.3. Let G be a CFMAXP graph with no separating triangle. Then:

- (a) if $\maxdeg G = 2$, $G = K_3$;
- (b) if $\maxdeg G = 3$, $G = K_4$;
- (c) if $\maxdeg G = 4$, G is the octahedron (cf. graph $G(6)$ in Figure 1);
- (d) and if $\maxdeg G = 5$, then G is one of the five graphs $G(7)$, $G(8)$, $G(9)$, $G(10)$ or $G(12)$ shown in Figure 1.

Proof. Parts (a) and (b) are trivial.

(c) Suppose $\maxdeg G = 4 = \deg v$. Let the 4 neighbors of v be u_1, u_2, u_3 and u_4 (in a clockwise orientation about v). Since G is MAXP, $u_1 \sim u_2 \sim u_3 \sim u_4 \sim u_1$ and the four corresponding triangles are faces. Moreover, again since G is MAXP, cycle $u_1 u_2 u_3 u_4 u_1$ is not a face boundary in G . So, without loss of generality, we may assume $\deg u_1 = 4$. Let w be the fourth neighbor of u_1 . Then by MAXP, $w \sim u_2$ and $w \sim u_4$. But now since $\deg u_2 = 4$, we have, by MAXP, that $w \sim u_3$ and hence G is the octahedron.

(d) Finally, suppose that $\maxdeg G = 5 = \deg v$. As before, let $\{u_1, \dots, u_5\}$ be the neighbors of v in a clockwise orientation about v . By MAXP, $u_1 \sim u_2 \sim u_3 \sim u_4 \sim u_5 \sim u_1$. As before, we may assume without loss of generality that $\deg u_1 \geq 4$.

Since G has no separating triangle, we may assume that $u_1 \not\sim u_3$ and $u_1 \not\sim u_4$. So let $w_1 \notin \{u_2, u_3, u_4, u_5\}$ be a fourth neighbor of u_1 . There are two cases to consider.

1. First suppose that $\deg u_1 = 4$. Then by MAXP, $w_1 \sim u_2$ and u_5 . If $\deg w_1 = 3$, then by MAXP, $u_2 \sim u_5$ and we get a separating triangle $vu_5 w_1 u_2 v$, a contradiction. So $\deg w_1 \geq 4$.

1.1 Suppose $\deg w_1 = 4$.

1.1.1. Suppose $w_1 \sim u_3$. Then, since G is 3-connected, triangle $w_1 u_3 u_2 w_1$ is a face and $\deg u_2 = 4$. Now $u_3 \not\sim u_5$ since G contains no separating triangle. But $\deg w_1 = 4$ then implies that G is not MAXP, a contradiction.

1.1.2. So suppose that $w_1 \not\sim u_3$. By symmetry, we may also assume that $w_1 \not\sim u_4$. So let x be the fourth neighbor of w_1 , $x \notin \{u_1, u_2, u_5\}$. By MAXP, $x \sim u_2, u_5$ and triangles $w_1 x u_2$ and $w_1 x u_5$ are face boundaries. Moreover, $\deg u_2 = \deg u_5 = 5$. Thus MAXP implies that $x \sim u_3, u_4$ and hence $\deg x = 5$. So triangles $u_2 x u_3$ and $u_5 x u_4$ are face boundaries and by 3-connectedness, $\deg u_3 = \deg u_4 = 4$. Thus we get graph $G(8)$ on 8 points.

1.2. So suppose $\deg w_1 = 5$.

1.2.1. Suppose $w_1 \sim u_3$.

Then since G has no separating triangle, triangle $w_1 u_2 u_3$ is a face and hence $\deg u_2 = 4$.

Now suppose that $w_1 \sim u_4$. Then $\deg w_1 = 5$ implies that triangle $w_1 u_5 u_4$ is a face as is triangle $w_1 u_3 u_4$. So G must be the 7 point graph $G(7)$.

So suppose that $w_1 \not\sim u_4$. Let z be the fifth neighbor of w_1 . Since $\deg w_1 = 5$ and G is MAXP, it follows that $z \sim u_5$ and hence $\deg u_5 = 5$, so triangle $w_1 u_5 z$ is a face as is triangle $z u_5 u_4$. But then by MAXP, it follows that $z \sim u_3$, $\deg u_3 = 5$ and triangle $z w_1 u_3$ is a face. Hence by 3-connectedness, G is the 8-point graph shown in Figure 2. But we have drawn and labeled it there so that it is obvious that it is isomorphic to graph $G(8)$ of Figure 1.



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Figure 2

1.2.2. So suppose that $w_1 \not\sim u_3$ (and by symmetry, that $w_1 \not\sim u_4$ as well). Let the fourth and fifth neighbors of w_1 be z_1 and z_2 . So we may assume by MAXP that $u_5 \sim z_1 \sim z_2 \sim u_2$. But then $\deg w_1 = 5$ implies that all triangles at w_1 are faces. Furthermore, we also have that $\deg u_2 = \deg u_5 = 5$. Hence MAXP implies that $z_1 \sim u_4$ and $z_2 \sim u_3$. Also triangles $z_1 u_5 u_4$ and $z_2 u_2 u_3$ are both faces.

Now suppose that $z_1 \sim u_3$. Then we must have that G is the 9 point graph $G(9)$ of Figure 1.

So suppose that $z_1 \not\sim u_3$ and by symmetry that $z_2 \not\sim u_4$. By MAXP, z_1 must have a fifth neighbor α . Again by MAXP, point $\alpha \sim z_2$ and triangle $\alpha z_1 z_2$ must be a face. So $\deg z_2 = 5$. Thus $\alpha \sim u_3, u_4$. So then $\deg u_3 = \deg u_4 = 5$ and G is the 10 point graph $G(10)$ of Figure 1.

2. So suppose that $\deg u_1 = 5$. By symmetry, we may also assume that $\deg u_2 = \deg u_3 = \deg u_4 = \deg u_5 = 5$ as well. Let the remaining 2 neighbors of u_1 be α_1 and β_1 in clockwise order about u_1 . So triangle $\alpha_1 \beta_1 u_1$ is a face boundary. The fifth neighbor of u_2 cannot be α_1, u_4 or u_5 since G contains no separating triangle. So let this fifth neighbor be α_2 where $\alpha_2 \notin \{v, u_1, \dots, u_5, \alpha_1, \beta_1\}$. Then we must have $\alpha_2 \sim \beta_1, u_3$. Let the fifth neighbor of u_3 be α_3 . Now $\alpha_3 \neq \beta_1, u_5$ since there are no separating triangles.

2.1. First suppose $\alpha_3 = \alpha_1$.

Now suppose $\deg \beta_1 = 4$. Then by MAXP, $\alpha_1 \sim \alpha_2$. So $\deg \alpha_1 = 5$ and then 3-connectedness implies that $\deg u_4 = 3$ and $\deg u_5 = 4$. Thus G is not MAXP, a contradiction.

So suppose that $\deg \beta_1 = 5$. Let γ be the fifth neighbor of β_1 . Then MAXP implies that $\gamma \sim \alpha_2$ and $\gamma \sim \alpha_1$. But then 3-connectedness implies that $\deg u_4 = 3$ and $\deg u_5 = 4$ and once again we contradict the hypothesis that G is MAXP.

2.2. So suppose that $\alpha_3 \neq \alpha_1$. Thus $\alpha_3 \notin \{v, u_1, \dots, u_5, \alpha_1, \beta_1, \alpha_2\}$. But then $\deg u_3 = 5$ and MAXP implies that $\alpha_3 \sim \alpha_2, u_4$.

2.2.1. Suppose $u_4 \sim \alpha_1$. By MAXP we must have $\alpha_1 \sim \alpha_3$ and hence $\deg \alpha_1 = 5$. But then again by MAXP we must have $\alpha_3 \sim \beta_1$ and $\deg \alpha_3 = 5$. Hence $\deg \beta_1 = 5$. So G is the graph shown in Figure 3 which is isomorphic to graph $G(10)$.

Figure 3

2.2.2. So suppose $u_4 \not\sim \alpha_1$.

2.2.2.1. Suppose $u_4 \sim \beta_1$. Then $\deg u_4 = \deg \beta_1 = 5$ and by 3-connectedness we have $\deg u_5 = 4$ and $\deg \alpha_1 = 3$. But this contradicts MAXP.

2.2.2.2. So we may assume that $u_4 \not\sim \alpha_1, \beta_1$. So let the fifth neighbor of u_4 be α_4 . Since $\deg u_4 = 5$ and since G is MAXP it follows that $\alpha_4 \sim \alpha_3, u_5$. So $\deg u_5 = 5$ and hence $\alpha_4 \sim \alpha_1$. (At this point, we have the graph shown in Figure 4.)

Figure 4

Now let us consider the possibilities for point α_1 .

If $\alpha_1 \sim \alpha_2$, then the degree of each is 5. But then by 3-connectedness, $\{\alpha_3, \alpha_4\}$ does not contain a cutset of G , and hence $\deg \alpha_4 = \deg \alpha_3 = 4$. But this contradicts MAXP.

If $\alpha_1 \sim \alpha_3$, we get a similar contradiction.

Next suppose that α_1 has a fifth neighbor α_5 , where $\alpha_5 \notin \{\beta_1, u_1, u_5, \alpha_4\}$. Then since $\deg \alpha_1 = 5$ it follows that $\alpha_5 \sim \beta_1, \alpha_4$. Then $\deg \beta_1 = 5$ and hence $\alpha_5 \sim \alpha_2$. Hence $\deg \alpha_2 = \deg \alpha_4 = 5$ and since G is MAXP, it follows that $\alpha_3 \sim \alpha_5$. So G must be the icosahedron labeled $G(12)$ in Figure 1.

So finally suppose that $\deg \alpha_1 = 4$. By symmetry, we may also suppose that $\deg \beta_1 = 4$. But then we contradict the fact that G is MAXP and the proof of the theorem is complete. ■

We now fix our attention on CFMAXP graphs which contain separating triangles.

The concepts of a *3/45-triangle* and a *3/45-nest* will prove central to our considerations. Any triangle (not necessarily a face boundary) in a CFMAXP graph G naturally separates the plane into two open regions R_1 and R_2 where, without loss of generality, we will call R_1 the *interior* of the triangle. Now let T_3 be such a triangle in CFMAXP graph G where

$V(T_3) = \{a_1, a_2, a_3\}$ and a_i sends i lines into region R_2 . We will call such a triangle T_3 an (interior) 345-triangle.

Now suppose $j \geq 0$ points of G lie interior to 345-triangle T_3 . If $j = 0$ then T_3 is a face boundary. In this case, denote the triangle, together with the half-lines in region R_2 incident with the three points of T_3 , by N_0 .

Next, suppose $j = 1$. Suppose a_4 is the only point of G interior to T_3 . Now since G is 3-connected, point a_4 is adjacent with all three points a_1, a_2 and a_3 . So by Theorem 2.1, $\deg a_3 = 6$ and triangles $a_3a_4a_1$ and $a_3a_4a_2$ are face boundaries. Since there are no other points interior to T_3 , triangle $a_4a_1a_2$ is also a face boundary. Denote this 4 point configuration including the 5 half-lines emanating from a_1, a_2 and a_3 into the exterior region R_2 by N_1 .

If $j = 2$, and a_4 and a_5 are the two interior points, then without loss of generality, we may assume that a_4 is adjacent to a_1, a_2 and a_3 and by Theorem 2.1, $\deg a_3 = 6$. Hence triangles $a_3a_4a_1$ and $a_3a_4a_2$ are face boundaries. So a_5 is interior to triangle $a_1a_2a_4$ and by 3-connectedness, $a_5 \sim a_1, a_2$ and a_4 . Hence all three triangles at a_5 are face boundaries. Denote the 5-point configuration (together with the 5 half-lines into region R_2) by N_2 .

Suppose $j = 3$ and that points a_4, a_5 and a_6 are interior to T_3 . Then without loss of generality, we may assume that $a_4 \sim a_1, a_2, a_3$, $a_5 \sim a_1, a_2$ and a_4 , that a_6 is interior to triangle $a_1a_4a_5$ and that $a_6 \sim a_1, a_4$ and a_5 . So we have 7 triangular faces interior to T_3 . Denote the resulting 6 point configuration (together with the half-lines into R_2) by N_3 .

In Figure 5, we display configurations N_0, N_1, N_2 and N_3 . (It is important to realize that the triangle $a_4a_5a_6$ is also a 345-triangle.)

Figure 5

Now we continue to define N_j 's, $j \geq 4$, inductively as follows. Suppose that points a_1, a_2, \dots, a_s , $s < j$, have been labeled so that a_s is adjacent to a_{s-1}, a_{s-2} and a_{s-3} . Then there remain $j - s > 0$ points interior to 345-triangle $T_s = a_s a_{s-1} a_{s-2}$. Since $\deg a_{s-2} \leq 6$ and T_s is a separating triangle, it follows that a_{s-2} is adjacent to exactly one point interior to T_s . Label this point a_{s+1} . Then $\deg a_{s-2} = 6$ and by MAXP we must have $a_{s+1} \sim a_s, a_{s-1}$ and triangles $a_{s+1} a_{s-1} a_{s-2}$ and $a_{s+1} a_s a_{s-2}$ must be face boundaries.

In this way, a unique labeling of all i points interior to T_3 is obtained and the structure of G interior to T_3 is completely determined. Call the subgraph of G induced by $\{a_1, \dots, a_j\}$ together with the half-lines from points a_1, a_2 and a_3 into region R_2 the (interior) 345-nest N_j based at a_1, a_2 and a_3 .

Now suppose interior 345-nest N_j forms part of a CFMAXP graph G . Let the unique exterior neighbor of a_1 be b_1 . Since line $a_1 b_1$ is the only line from a_1 to the exterior region R_2 , it must be the case that $b_1 \sim a_2, a_3$ and triangles $b_1 a_1 a_2$ and $b_1 a_1 a_3$ must be face boundaries. Let b_2 be the second neighbor of a_2 in R_2 . Then since $a_2 b_1$ and $a_2 b_2$ are the only two lines from a_2 into R_2 , triangles $a_2 b_1 b_2$ and $a_2 b_2 a_3$ must be face boundaries. Finally, let b_3 be the third point in R_2 adjacent to a_3 . Then $b_3 \sim b_1, b_2$ and triangles $b_3 b_1 a_3$ and $b_3 b_2 a_3$ are also face boundaries. (See Figure 6.)

Figure 6

Moreover, we now see that the three points of triangle $b_1 b_2 b_3$ send 3, 2 and 1 lines respectively into region R_1 . Thus we may call triangle $b_1 b_2 b_3 = T'_3$ an (exterior) 345-triangle. Clearly, if $k \geq 0$ points of G lie in R_2 , but exterior to T'_3 , we can repeat our argument about the interior of triangle T_3 to conclude that the k points of G exterior to T_3 can be labeled b_4, b_5, \dots, b_{k+3} so that $b_4 \sim b_1, b_2$ and b_3 , $b_5 \sim b_1, b_2$ and $b_4, b_6 \sim b_1, b_4$ and b_5 and for $k \geq 4$, $b_{k+3} \sim b_{k+2}, b_{k+1}$ and b_k . We call the resulting configuration (together with the 6 half-lines from b_1, b_2 and b_3 into the interior of triangle $b_1 b_2 b_3 = T'_3$) an exterior nest N'_k . Finally, we call graph G the amalgamation of nests N_j and N'_k at triangle $T_3 = a_1 a_2 a_3$ and write G as $N_j \odot N'_k$.

We have thus proved the first half of the following theorem.

Theorem 2.4. Given any CFMAXP graph G with a separating (interior) 345-triangle $T_3 = a_1a_2a_3$, G can be expressed as the amalgamation of 2 nests $N_j \odot N'_k$ at T_3 .

Conversely, the amalgamation of 2 nests $N_j \odot N'_k$ at a 345-triangle T_3 , $j \geq 1$, $k \geq 0$ results in a CFMAXP graph.

Proof. To prove the second half of this theorem, we need only check that $N_j \odot N'_k$ is claw-free. Moreover, by MAXP, we need only check for claws at points of degree 6. Finally, by symmetry, it suffices to check only those points of degree 6 in the nest N_j and therefore in the graph $G_j = N_j \odot N'_0$, for $j \geq 1$.

It is easy to check by hand that G_0, G_1, G_2 and G_3 have no claws. (They have 0, 1, 2 and 3 points of degree 6 respectively.)

Now suppose G_r has no claws for $3 \leq r < i$ and consider graph G_{r+1} . Graph G_{r+1} is obtained from G_r by inserting one new point a_{r+4} inside triangle $a_{r+3}a_{r+2}a_{r+1}$ in G_r and joining a_{r+4} to each of these 3 points. The only newly formed point of degree 6 is a_{r+1} which is adjacent to a_r, a_{r-1} and a_{r-2} by definition of G_{r-2}, G_{r-1} and G_r , as well as to a_{r+2}, a_{r+3} and a_{r+4} . Now in G_{r+1} , there are 6 triangular faces at point a_{r+4} . In addition, $a_{r+3} \sim a_{r+2}$ by definition of G_r and $a_{r-1} \sim a_{r-2}$ by definition of G_{r-3} . It thus follows that there are no claws at a_{r+1} .

Thus G_{r+1} , and hence by induction, all G_j 's are claw-free. ■

Corollary 2.5. Let G be CFMAXP with a separating 345-triangle. Then G contains precisely 2 points of degree 3.

Proof. From the preceding theorem we can write $G = N_j \odot N'_k$ and each of the 2 nests contains exactly 1 point of degree 3. ■

Now with the idea of separating 345-triangles in mind, we can proceed with our characterization of CFMAXP graphs.

Theorem 2.6. Suppose graph G is CFMAXP with a separating triangle, but no separating 345-triangle. Then G must be one of the 7 graphs displayed in Figure 7.

Proof. Let $T = abc$ be a separating triangle. From Theorem 2.1 we know that $\maxdeg G \leq 6$. Moreover, by 3-connectedness, we know that each of the points a, b and c sends at least one line interior to T and at least one line exterior to T . We proceed to check all possibilities. For $x = a, b$ and c , let us denote by m_x and n_x the number of lines from x into the interior of T and into the exterior of T respectively.

1. $m_a = m_b = m_c = 1$.

1.1. Suppose $n_a = n_b = n_c = 1$. There must be a neighbor u_1 of both a and b in the interior of T such that triangle abu_1 is a face. Then $c \sim u_1$ too and all triangles at u_1 must be face boundaries. Similarly in the exterior of T . Thus G must be isomorphic to $G(5)$ in Figure 7.

1.2. Suppose $n_a = 2$ and $n_b = n_c = 1$. Let the exterior neighbors of a be u_1 and u_2 . Then $\deg a = 5$ and by MAXP, $u_1 \sim u_2 \sim b$ and $u_1 \sim c$. So all triangles $au_1u_2, au_2b, abv, avc, acu_1$ must be face boundaries. Since G is MAXP, there must be a

point u_3 exterior to the quadrilateral $b c u_1 u_2$. But then $\{u_1, u_2\}$ must contain a cut in G , contrary to 3-connectedness.

1.3. Suppose $n_a = 3$ and $n_b = n_c = 1$. Let the exterior neighbors of a be u_1, u_2 and u_3 . Then $\deg a = 6$, so $c \sim u_1 \sim u_2 \sim u_3 \sim b$ and we must have 6 triangular faces at a . Since there is no claw at a , $u_1 \sim u_3$. Also since $\deg b = 4$, triangle $b v c$ must be a face. But then since quadrilateral $b c u_1 u_3$ is not a face, $\{u_1, u_3\}$ must contain a cut in G , contradicting 3-connectedness.

1.4. Suppose $n_a = n_b = 2$ and $n_c = 1$. Again let u_1 and u_2 be the exterior neighbors of a . Again we must have $c \sim u_1 \sim u_2 \sim b$ and 5 triangular faces at point a . Also, since $\deg c = 4$, triangle $c v b$ is also a face.

1.4.1. Suppose the second exterior neighbor of b is u_1 . Then triangle $b u_1 u_2$ must be a face as must triangle $b c u_1$. so we obtain graph $G(6a)$ shown in Figure 7.

1.4.2. So suppose the second exterior neighbor of b is point u_3 , $u_3 \neq u_1$. Then $\deg b = 5$ and by MAXP we have $u_2 \sim u_3$ and $u_3 \sim c$. But this contradicts the assumption that $n_c = 1$.

1.5. Suppose $n_a = 3$, $n_b = 2$ and $n_c = 1$. Let the remaining three exterior neighbors of a be u_1, u_2 and u_3 labeled so that the clockwise order of all 5 neighbors of a is b, c, u_1, u_2, u_3 . Then since $\deg b = 6$, we must have $c \sim u_1 \sim u_2 \sim u_3 \sim b$. Since there is no claw at a , points u_1 and u_3 must be adjacent. So there are 6 triangular faces at point a and since $\deg c = 4$, it follows that triangle $c v b$ is also a face. Since G is MAXP, it follows that $u_1 \sim b$ and hence triangle $u_1 b c$ is a face. Furthermore, $\deg b = 5$ implies that there are no more points outside triangle $b u_1 u_3$; i.e., triangle $b u_1 u_3$ is also a face boundary.

Now if there are any points exterior to triangle $u_1 u_2 u_3$, then triangle $u_1 u_2 u_3$ is a separating 345-triangle, contrary to assumption. So triangle $u_1 u_2 u_3$ is a face boundary as well. So graph G has 7 points, but contains a separating 345-triangle, namely $u_1 u_3 a$, contrary to assumption.

1.6. Suppose $n_a = n_b = 3$ and $n_c = 1$.

So $c \sim u_1 \sim u_2 \sim u_3 \sim b$ and we have 6 triangular faces at a . Also, since there is no claw at a , we have $u_1 \sim u_3$. Moreover, $b \sim u_1$ and triangle $b c u_1$ is a face, since $\deg c = 4$ and G is MAXP. Let u_4 be the sixth neighbor of b . Then u_4 is in the interior of triangle $u_1 u_3 b$. Since $\deg b = 6$, we have $u_4 \sim u_3, u_1$ and six triangular faces at point b . But then $G[u_1, c, u_2, u_4]$ is a claw, contrary to hypothesis.

1.7. Suppose $n_a = n_b = n_c = 2$.

Let u_1 and u_2 be the two exterior neighbors of a . Then $c \sim u_1 \sim u_2 \sim b$ and there are 5 triangular faces at a . Since G is planar, there exists a point u_3 exterior to the quadrilateral $b c u_1 u_2$ such that either b or c is adjacent to u_3 . By symmetry, without loss of generality, we may suppose that $b \sim u_3$. Since $\deg b = 5$, $c \sim u_3 \sim u_2$ and there must be 5 triangular faces at b . Since $\deg c = 5$, $u_1 \sim u_3$ and triangle $u_1 c u_3$ is a face boundary.

1.7.1. Suppose $\deg u_3 = 4$.

Then triangle $u_1 u_2 u_3$ is a face boundary and G must be the 7 point graph $G(7a)$.

1.7.2. Suppose $\deg u_3 = 5$.

Then let u_4 be the fifth neighbor of u_3 . Since $\deg u_4 = 5$ it follows that $u_4 \sim u_1, u_2$ and there are 5 triangular faces at u_3 .

1.7.2.1. Suppose $\deg u_4 = 3$.

Then G must be the 8 point graph $G(8a)$.

1.7.2.2. Suppose $\deg u_4 = 4$.

Let u_5 be the fourth neighbor of u_4 . Then, since $\deg u_4 = 4$, it follows that $u_5 \sim u_1, u_2$ and that there are 4 triangular faces at u_4 . But then $G[u_1, u_4, u_3, u_5]$ is a claw, a contradiction.

1.7.2.3. Suppose $\deg u_4 = 5$.

Let the 2 remaining neighbors of u_4 be u_5 and u_6 . Since $\deg u_4 = 5$, we may suppose that $u_5 \sim u_1, u_6, u_6 \sim u_2$ and that there are 5 triangular faces at u_4 . Moreover, since G is MAXP, there exists a point w in the exterior of quadrilateral $u_1u_2u_5u_6$. But then $\deg u_1 = \deg u_2 = 6$ implies that $\{u_5, u_6\}$ contains a cutset of G , contradicting 3-connectedness.

1.7.2.4. Finally, suppose $\deg u_4 = 6$.

Let u_5, u_6, u_7 be the 3 remaining neighbors of u_4 . Then we may suppose that $u_1 \sim u_5 \sim u_6 \sim u_7 \sim u_2$ and there are 6 triangular faces at u_4 . Since there is no claw at u_4 , points u_5 and u_7 are adjacent. But then since $\{u_5, u_7\}$ is not a cutset and since $\deg u_1 = \deg u_2 = 6$, quadrilateral $u_1u_2u_7u_5$ must be a face boundary, contrary to the MAXP hypothesis.

1.7.3. So we may suppose $\deg u_3 = 6$.

Let w_4 and w_5 be the remaining 2 neighbors of u_3 . Then we may suppose that $w_4 \sim u_1, w_5, w_5 \sim u_2$ and that there are 6 triangular faces at u_3 . Since there is no claw at u_3 , it follows that $w_5 \sim u_1$. So $\deg u_1 = 6$ and there are 6 triangular faces at point u_1 . But then $G[u_3, u_1, u_2, w_4]$ is a claw at u_3 , a contradiction.

1.8. Suppose $n_a = 3$ and $n_b = n_c = 2$.

As before, since $\deg a = 6$, it follows that $c \sim u_1 \sim u_2 \sim u_3 \sim b$ and there are 6 triangular faces at a . Moreover, $u_1 \sim u_3$ since there is no claw at a .

1.8.1. Suppose $b \sim u_1$.

Since $\deg b = 5$, triangle bu_1u_3 must be a face boundary. Suppose $\deg u_1 = 5$. Then triangle u_3u_1a is a separating 345-triangle, contrary to hypothesis. So $\deg u_1 = 6$. Let u_4 be the sixth neighbor of u_1 . If u_4 is interior to triangle $u_1u_3u_2$, triangle u_1u_3a is again a separating 345-triangle. So u_4 is exterior to triangle u_1bc and hence triangle $u_1u_2u_3$ is a face boundary. On the other hand, $\deg u_1 = 6$ implies that $u_4 \sim b, c$. But then $G[b, u_1, u_3, u_4]$ is a claw, a contradiction.

1.8.2. So we may suppose that b and u_1 are not adjacent.

By symmetry, we may also suppose that c and u_3 are not adjacent as well. Let w be the fifth neighbor of b . Since $m_a = m_b = m_c = 1$, point w must lie outside quadrilateral bcu_1u_3 . But then $\deg b = 5$ implies that $w \sim c, u_3, u_1$ and that there are 5 triangular faces at b and at c .

1.8.2.1. Suppose $\deg u_1 = 5$.

Then triangles $u_1u_2u_3$ and u_1u_3w must be face boundaries and G must be the 8 point graph $G(8b)$.

1.8.2.2. So suppose $\deg u_1 = 6$.

Let x be the sixth neighbor of u_1 . If x is interior to triangle $u_1u_2u_3$, then $G[u_1, x, a, w]$ is a claw, while if x is exterior to triangle $u_1u_2u_3$, then $G[u_1, x, u_2, c]$ is a claw.

1.9. Suppose $n_a = n_b = 3$ and $n_c = 2$. Once again, let u_1, u_2 and u_3 be the 3 neighbors of a exterior to triangle abc (where we shall assume that u_1, u_2, u_3, b and c are

in clockwise order about point a). Then $c \sim u_1 \sim u_2 \sim u_3 \sim b$ and there are 6 triangular faces at a . Moreover, since there is no claw at a , points u_1 and u_3 must be adjacent.

1.9.1. Suppose $b \sim u_1$.

If w is exterior to triangle u_1bc , then $G[b, w, v, u_3]$ is a claw at b . So w is interior to triangle u_1u_3b . So $\deg b = 6$ and by MAXP it follows that $w \sim u_3$ and there are 6 triangular faces at b . But this contradicts the assumption that $n_c = 2$.

1.9.2. So suppose $b \not\sim u_1$ (and by symmetry, that $c \not\sim u_2$).

So let w_1 and w_2 be the two neighbors of b outside quadrilateral $b u_1 u_3$. Since $\deg b = 6$, by MAXP we may assume that $w_1 \sim c$ and $w_2 \sim u_3$. Moreover, since there are 6 triangular faces at b , it must also be the case that $w_1 \sim w_2$ and triangle $b w_1 w_2$ is one of these faces. Since there is no claw at b , points w_1 and u_3 are adjacent. So $\deg u_3 = 6$ and therefore, triangles $u_1 u_2 u_3$ and $u_3 w_1 w_2$ are also face boundaries. Finally, also by MAXP, it must be that $u_1 \sim w_1$ and the triangle $u_1 u_3 w_1$ must be the boundary of the infinite face at u_3 .

Now if $\deg u_1 = 5$, then triangle $u_1 w_1 c$ is a face boundary and G is the 9 point graph $G(9a)$. So suppose that $\deg u_1 = 6$. Let the sixth neighbor of u_1 be w_3 . Then w_3 must lie in the interior of triangle $u_1 c w_1$. But then by MAXP, $w_3 \sim c$, contradicting the assumption that $n_c = 2$.

1.10. Suppose $n_a = n_b = n_c = 3$.

Again, let the six neighbors of a , in a clockwise orientation, be b, v, c, u_1, u_2 and u_3 . As before, $b \sim u_1 \sim u_2 \sim u_3 \sim b$, there are six triangular faces at point a and since there is no claw at a , points u_1 and u_3 are adjacent.

1.10.1. Suppose $b \sim u_1$.

Let w_1 be the sixth neighbor of point b . If w_1 lies in the exterior of triangle u_1bc , then $G[b, w_1, v, u_3]$ is a claw. So w_1 must lie in the interior of triangle u_1u_3b . Since $\deg b = 6$ and by MAXP we have $w_1 \sim u_3$, $w_1 \sim u_1$ and triangles $w_1 b u_1$ and $w_1 u_3 u_1$ must be face boundaries. But then $G[u_1, u_2, w_1, c]$ is a claw at u_1 .

1.10.2. So suppose $b \not\sim u_1$ (and by symmetry, $c \not\sim u_3$).

Let w_1 and w_2 be the remaining 2 neighbors of b so that in a clockwise order, the 6 neighbors of b are a, u_3, w_2, w_1, c and v . Then $c \sim w_1 \sim w_2 \sim u_3$ and there are 6 triangular faces at b . Since there is no claw at b , $w_1 \sim u_3$. Also since $\deg u_3 = 6$, it follows that triangle $u_3 w_2 w_1$ is a face boundary, $u_1 \sim w_1$ and triangle $u_1 u_3 w_1$ is a face boundary, and finally, that triangle $u_1 u_2 u_3$ is a face boundary.

Now let w_3 be the sixth neighbor of c . Then w_3 is interior to triangle $u_1 c w_1$. Also $\deg c = 6$ implies that $w_3 \sim u_1, w_1$, that there are 6 triangular faces at c and hence also at u_1 . So G must be the 10 point graph $G(10a)$. This completes Case 1.

2. Suppose $m_a = 2$ and $m_b = m_c = 1$.

Let the interior neighbors of a be u_1, u_2 . Then we may suppose that $c \sim u_1 \sim u_2 \sim b$ and triangles $a c u_1, a u_1 u_2$ and $a u_2 b$ must be face boundaries. However, since $\{u_1, u_2\}$ does not contain a cutset of G , it follows that quadrilateral $b c u_1 u_2$ is a face of G . But this contradicts the hypothesis that G is MAXP.

3. Suppose $m_a = 3$ and $m_b = m_c = 1$.

Arguing in a manner similar to that in Case 2, it is easily seen that the interior face which contains line bc in its boundary cannot be a triangle. So once again, MAXP is

contradicted.

4. Suppose $m_a = m_b = 2$ and $m_c = 1$.

Let u_1 and u_2 be the 2 interior neighbors of a so that $c \sim u_1 \sim u_2 \sim b$ and triangles acu_1 , au_1u_2 and au_2b must be face boundaries. Also since $m_c = 1$, it follows that $b \sim u_1$ and triangle u_1bc is a face boundary.

4.1. Suppose $n_a = 1$ and $n_b = n_c = 2$.

But then triangle u_1ab must be a separating 345-triangle which is a contradiction.

4.2. Suppose $n_a = 1$, $n_b = 2$ and $n_c = 3$.

Then triangle abc is a separating 345-triangle.

4.3. Suppose $n_a = n_b = 2$ and $n_c = 1$.

Let w_1, w_2 be the 2 exterior neighbors of a . Since $\deg a = 6$ and G is MAXP, it follows that $b \sim w_2 \sim w_1 \sim c$ and there are 6 triangular faces at a . Points b and w_1 are adjacent since $\deg c = 4$. But then $G[b, w_2, u_2, c]$ is a claw at b .

4.4. Suppose $n_a = n_b = n_c = 2$.

Let w_1 and w_2 be as in Case 4.3. Again we have that $c \sim w_1 \sim w_2 \sim b$ and G has 6 triangular faces at a . Since there is no claw at b , points c and w_2 must be adjacent. Moreover, since $\deg c = 5$, triangles cw_1w_2 and bcw_2 must be face boundaries. Hence we obtain a 7 point graph in which triangle w_2ca is a separating 345-triangle.

4.5. Suppose $n_a = n_b = 2$ and $n_c = 3$.

Again let w_1 and w_2 be as in Case 4.3. As before, $c \sim w_1 \sim w_2 \sim b$ and we have 6 triangular faces at a .

4.5.1. Suppose $b \sim w_1$.

Then $\deg b = 6$ implies that there are 6 triangular faces at b . Hence $\{w_1, c\}$ must contain a cutset of G contradicting 3-connectedness.

4.5.2. So suppose $b \not\sim w_1$.

Let w_3 be the second exterior neighbor of b . Since $\deg b = 6$, we must have $c \sim w_3$ and $w_2 \sim w_3$. Also, since there is no claw at b , it follows that $c \sim w_2$. But then $G[c, u_1, w_1, w_3]$ is a claw at c .

It is straightforward to see that, due to symmetry, there remains only one additional case to treat.

5. Suppose $m_a = m_b = m_c = n_a = n_b = n_c = 2$.

Let u_1 and u_2 be the 2 internal neighbors of a . Since $\deg a = 6$ by MAXP we have that $c \sim u_1 \sim u_2 \sim b$ and triangles acu_1 , au_1u_2 and au_2b are face boundaries.

Suppose $b \sim u_1$. Then triangle bu_1c is a face boundary and hence $m_c = 1$, contrary to assumption. So $b \not\sim u_1$. By symmetry we may also assume that $c \not\sim u_2$ as well. So let u_3 be the second interior neighbor of b . Since $m_b = 2$, we have $u_2 \sim u_3 \sim c$ and hence both triangles bu_2u_3 and bu_3c are face boundaries. Moreover, since $m_c = 2$, it follows that $u_1 \sim u_3$ and that triangle cu_1u_3 is a face boundary as well.

Now let w_1 and w_2 be the 2 exterior neighbors of a . Since $\deg a = 6$ we may assume that $c \sim w_1 \sim w_2 \sim b$ and that there are 6 triangular faces at a . Since there is no claw at a , points w_1 and b must be adjacent. But then $G[a, w_2, u_2, c]$ is a claw, contradicting the hypothesis and completing the proof of the theorem. ■

Figure 7

3. Traversability in CFMAXP Graphs

Historically, the first theorem about Hamilton cycles in MAXP graphs seems to be the following classical result due to Whitney [15].

Theorem 3.1. If G is MAXP with no separating triangle, then G has a Hamilton cycle. ■

As is customary, let us denote the set of all points adjacent to a point v by $N(v)$ and call the induced subgraph $G[N(v)]$ the neighborhood graph of v in G . Graph G is said to be locally n -connected if for all $v \in V(G)$, $G[N(v)]$ is n -connected.

In order to present some more recent results on traversability in MAXP graphs, we shall need the following easy lemma relating MAXP and local n -connectivity.

Lemma 3.2. Let G be a connected planar graph with $|V(G)| \geq 4$. Then G is MAXP if and only if G is locally 2-connected.

Proof. If G is MAXP and $v \in V(G)$, then $G[N(v)]$ is a cycle and hence 2-connected.

To prove the converse, let us suppose that G is locally 2-connected, but has a face $F = u_1 \cdots u_k$ of size $k \geq 4$. Consider $N(u_1)$. Since G is 3-connected, we know that $N(u_1)$ contains u_2, u_k and at least one other point. If $u_1 \sim u_j$, for some j , $2 < j < k$, then $\{u_1, u_j\}$ is a 2-cut in G contradicting 3-connectedness. So $N(u_1) \cap \{u_1, \dots, u_k\} = \{u_2, u_k\}$.

Let v be a third neighbor of u_1 , $v \notin \{u_2, u_k\}$, and let $G[N(v)]$ be denoted by G_v . Since G_v is 2-connected, there is a cycle Z through points u_2 and u_k where $Z \subseteq G_v$ and hence Z contains only 2 points of the boundary of face F , namely u_2 and u_k . Now cycle Z can be thought of as the union of two openly disjoint paths $P_1 \cup P_2$ where each path P_i joins u_2 to u_k , but $V(P_1) \cap V(P_2) = \{u_2, u_k\}$.

Suppose each P_i contains at least 3 points. Then $P_1 \cup P_2 \cup \{u_1 u_2, u_k u_1\} \cup (\partial F - u_1)$ is a homeomorph of the complete bipartite graph $K_{2,4}$ and since F is a face, relabeling P_1 and P_2 if necessary, we may suppose, without loss of generality, that cycle $P_2 \cup (\partial F - u_1)$ separates any point on $P_1 - u_2 - u_k$ from u_1 . But this contradicts the fact that v is a neighbor of u_1 . So $P_1 - u_2 - u_k = \emptyset$, that is, P_1 is just the single line $u_2 u_k$. But then $\{u_2, u_k\}$ is a 2-cut in G separating u_1 from u_3 contradicting the 3-connectedness of G . ■

A graph G is panconnected if for each pair of distinct points u and v in G and for every integer m , $d(u, v) \leq m \leq |V(G)| - 1$, there is a path joining u and v of length m . A graph is Hamiltonian connected if each pair of distinct points is joined by a spanning (i.e., Hamiltonian) path. A graph is line-Hamiltonian if each line lies on a Hamilton cycle. A graph G is point-pancyclic if for all points $v \in V(G)$ and all integers m , $3 \leq m \leq |V(G)|$, there is a cycle of length m containing point v .

It was pointed out by Clark [1] that panconnected \Rightarrow Hamiltonian connected \Rightarrow line-Hamiltonian \Rightarrow Hamiltonian and panconnected \Rightarrow point-pancyclic \Rightarrow Hamiltonian.

The following result is due to Kanetkar and Rao (Theorem 4 of [4]).

Theorem 3.3. If G is connected, locally 2-connected and claw-free, then G is pan-connected. ■

Using this result, together with the observations of Clark, our next result is immediate.

Corollary 3.4. If G is CFMAXP, then G is panconnected, Hamiltonian-connected, line-Hamiltonian, point-pancyclic and Hamiltonian. ■

4. Matching in CFMAXP Graphs

A graph G on p points is bicritical if $G - u - v$ has a perfect matching for all pairs of distinct points u and v in G . Such graphs play an important role in a canonical theory of the decomposition of graphs in terms of their maximal (or perfect) matchings. (Cf. Lovász and Plummer [6].)

Another concept closely related to bicriticality is that of n -extendability. (The concept was introduced for graphs in general in [10] and later studied in the special case of planar graphs in [11] and [3].) Let p and n be positive integers and suppose $n \leq (p-2)/2$. A graph G is said to be n -extendable if G has a matching of size n and every matching of size n extends to (i.e., is a subset of) a perfect matching. If G is not bipartite, then the following two implications hold. G is 2-extendable $\Rightarrow G$ is bicritical $\Rightarrow G$ is 1-extendable. The first implication follows from Theorem 4.2 of [10] and the second implication is immediate from the definition of bicritical.

Graphs which are CFMAXP can be categorized nicely with respect to the concepts of 2-extendability and bicriticality. To wit, we have the following result.

Theorem 4.1. If G is CFMAXP and $|V(G)| \geq 4$ and even, then:

- (a) G is bicritical and
- (b) G either is not 2-extendable, or else is the icosahedron (which is 2-extendable).

Proof. Let u and v be any 2 points in G . Then by Corollary 3.4 there is a Hamiltonian path P joining u and v in G . Denote this path by $P = (u = u_1)u_2 \cdots u_{2k-1}(u_{2k} = v)$. Since P is of odd length, so is subpath $P' = P - u - v = u_2 \cdots u_{2k-1}$. But then $M = \{u_2u_3, u_4u_5, \dots, u_{2k-2}u_{2k-1}\}$ is a perfect matching for $G - u - v$ and hence G is bicritical.

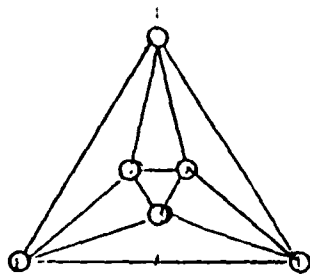
On the other hand, it is easy to see that no CFMAXP graph, other than the icosahedron, can be 2-extendable. Let G be CFMAXP. If G has no separating 345-triangle, appealing to Theorems 2.3 and 2.6, we see that there are only 15 graphs to check and it is easy to see that the only one of these which is 2-extendable is the icosahedron.

Now suppose that G contains a separating 345-triangle $T = abc$. Then it contains such a triangle with exactly 1 point on its interior. Let this interior point be d . Now let e be a fifth point of G where e is exterior to the triangle $T = abc$, but adjacent to one of the points a, b or c . Say, without loss of generality, that e is adjacent to a . Then clearly the 2 lines bc and ae do not extend to a perfect matching. ■

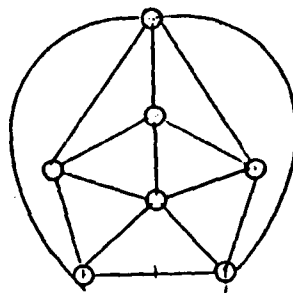
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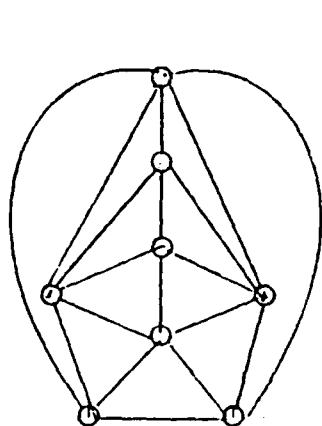
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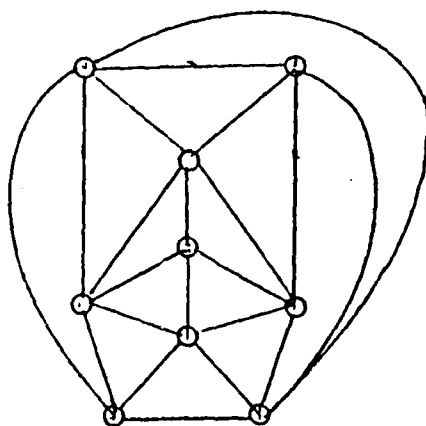
G(6)



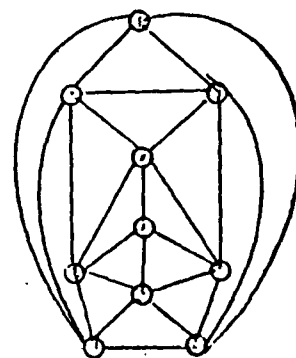
G(7)



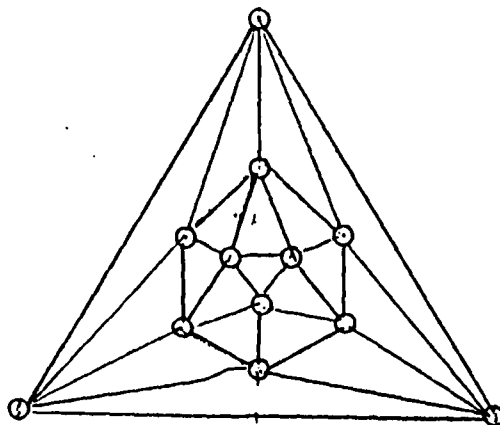
G(8)



G(9)



G(10)



G(12)

Figure 1

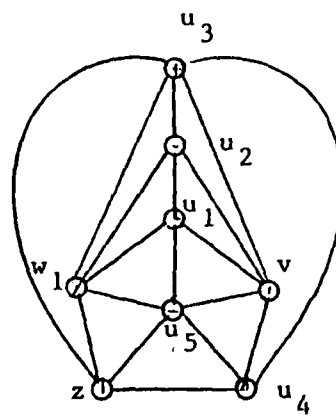


Figure 2

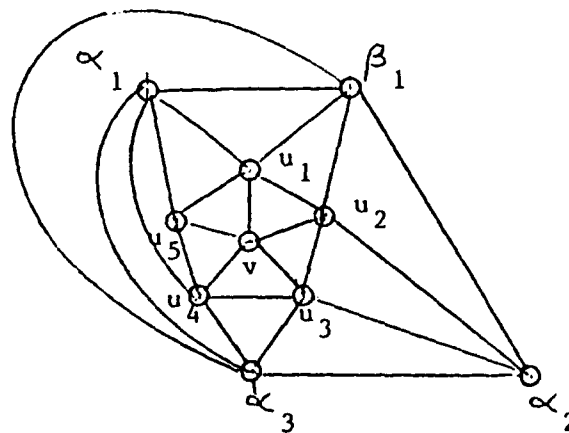


Figure 3

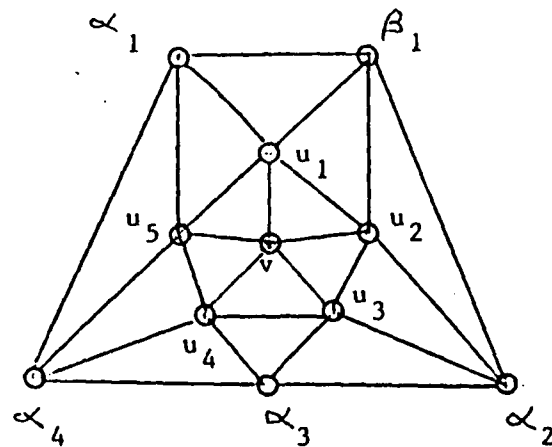


Figure 4

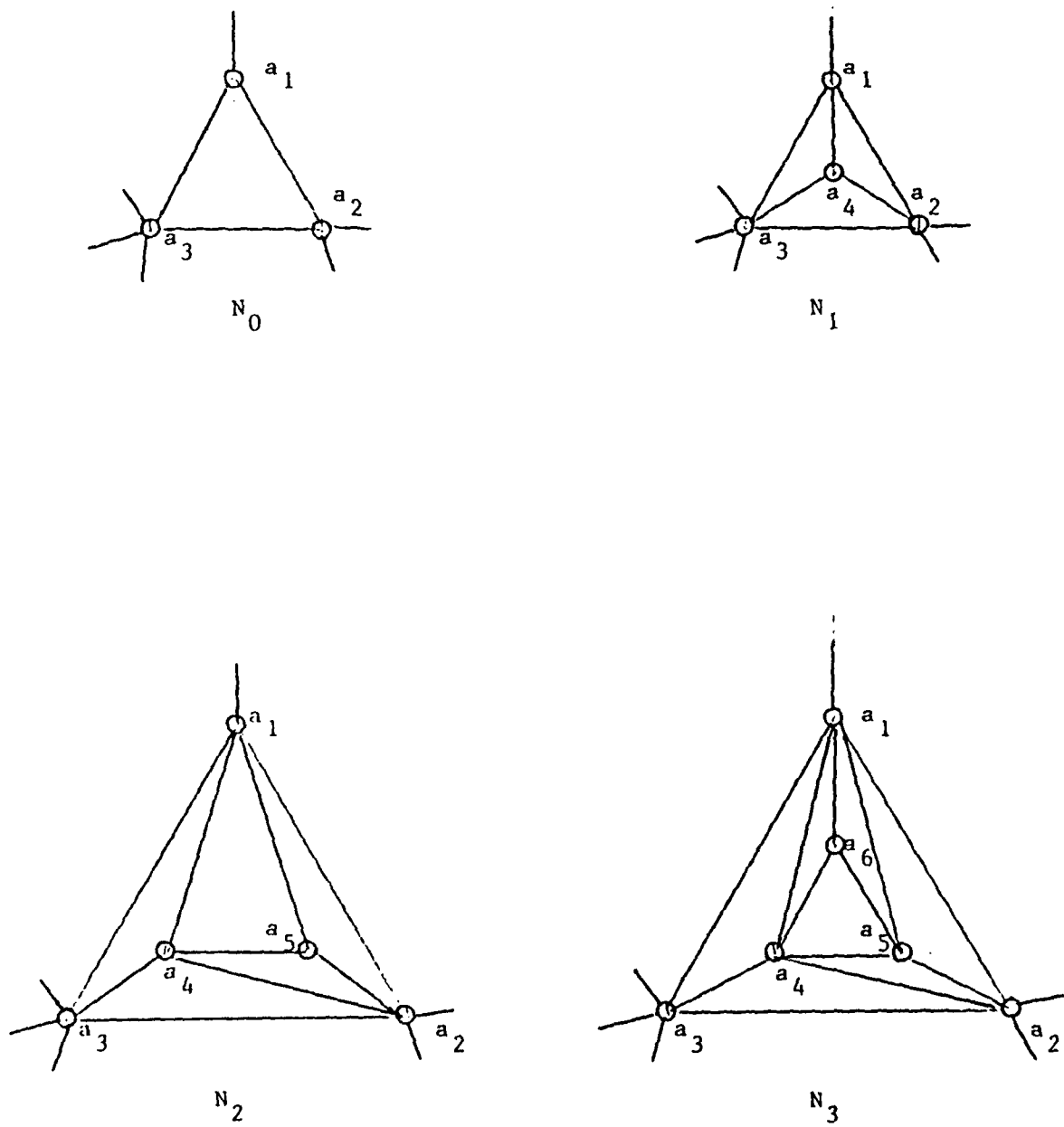


Figure 5

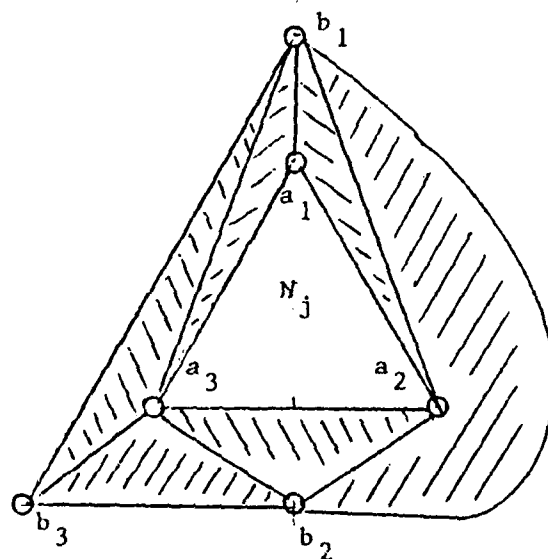
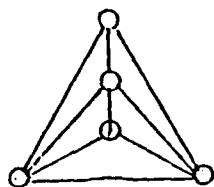
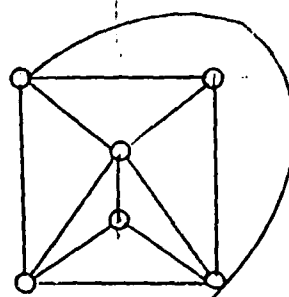


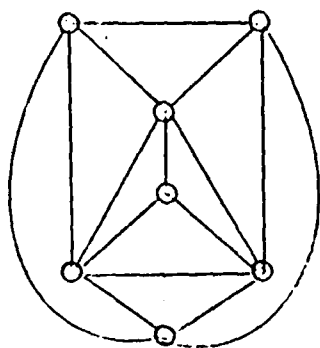
Figure 6



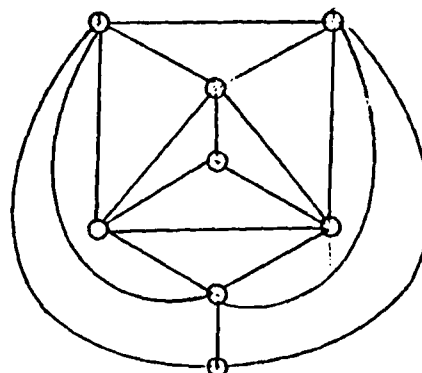
G(5)



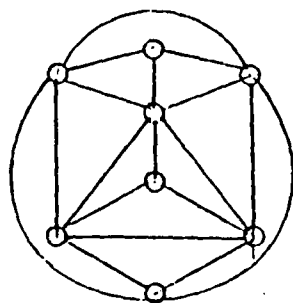
G(6a)



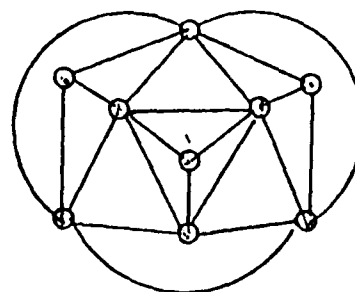
G(7a)



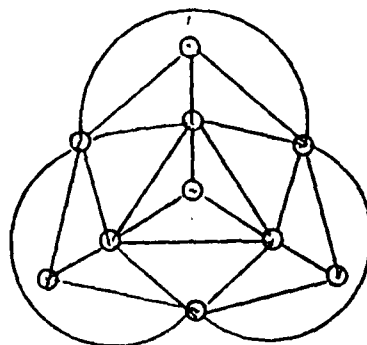
G(8a)



G(8b)



G(9a)



G(10a)

Figure 7